



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# A VARIABLE SYSTEM OF SEVENS ON TWO TWISTED CUBIC CURVES

By H. S. White

DEPARTMENT OF MATHEMATICS, VASSAR COLLEGE

Received by the Academy, May 15, 1916

Seven points chosen at random on a twisted cubic curve, like six points on a conic in the plane, give rise to a distinctive theorem; for as five points determine a conic, so the twisted cubic is determined by six points. In the case of the conic, this is the theorem of the Pascal hexagon, six points in a definite order leading to a definite line. Conic and line remaining fixed, the hexagon may vary with four degrees of freedom. In the case of the twisted cubic, not a mere sequence of the seven points, but an arrangement of them in seven triads, determines seven planes, and the theorem states that these planes are all osculated by a second twisted cubic curve. So much was established by a direct proof in these PROCEEDINGS in August, 1915; but the question of variability, whether the points and planes are free to move while the two curves remain fixed, was not examined. Now it is found that *the system is variable with one degree of freedom*. Full proof is contained in a paper soon to appear in the *Transactions of the American Mathematical Society*. The following is an outline.

Every twisted cubic  $C_3$  is a rational curve, and the homogeneous coordinates of its points are cubic functions of a variable parameter:

$$x_1 = f_1(\lambda), x_2 = f_2(\lambda), \dots, x_4 = f_4(\lambda).$$

In the same way the osculating planes of any second cubic curve  $K_3$  are represented by cubic functions of a second parameter:

$$u_1 = g_1(\mu), u_2 = g_2(\mu), \dots, u_4 = g_4(\mu).$$

Every point of the first may be put in relation to the three planes of the other that pass through it by the equation

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0,$$

or

$$\sum_1^4 f_i(\lambda) \cdot g_i(\mu) = 0,$$

of the third degree in  $\lambda$  and also in  $\mu$ . Conversely, every bicubical relation, or (3,3) correspondence, may be interpreted as such a point-to-plane relation between points of an arbitrary cubic  $C_3$  and osculat-

ing planes of a second. This second curve  $K_3$ , however, is completely determined by the first,  $C_3$ , and the relation

$$F(\lambda, \mu) = \sum f_i(\lambda) \cdot g_i(\mu) = 0;$$

for there is only one way of expressing a given bicubical function  $F(\lambda, \mu)$  linearly in terms of four independent cubics  $f_1(\lambda), f_2(\lambda), f_3(\lambda), f_4(\lambda)$ .

Such a bicubical function  $F(\lambda, \mu)$  contains 15 arbitrary coefficients, besides one multiplicative constant. If it relates cubic curves whose relative situation is like that of the  $C_3$  and  $K_3$  mentioned above, we may say that the relation  $F(\lambda, \mu)$  admits a solution of period 7, or briefly, that it admits a  $\Delta_7$ . A special kind of (3, 3) relation is that which factors into three (1, 1) relations or projectivities:

$$F(\lambda, \mu) \equiv \varphi(\lambda, \mu) \cdot \psi(\lambda, \mu) \cdot \chi(\lambda, \mu),$$

the triply bilinear relation. Of this special kind there is a sub-species which admits a  $\Delta_7$ .

To prove the theorem stated above, viz., *that every (3, 3) relation which admits one  $\Delta_7$  must necessarily admit a simple infinity of  $\Delta_7$ 's*, I proceed by counting the number of free constants in each of the four classes of (3.3) relations which have just now been noticed.

The first class, the general (3, 3) relation, contains 15 constants, including 3 that might have been deducted for linear transformation of either  $\lambda$  or  $\mu$ . The second class, that admitting one  $\Delta_7$  (or more than one), contains the three constants of a linear transformation and apparently 7 others, since according to the former theorem cited above the 7 points on the  $C_3$  can be chosen at random. If however the presence of one  $\Delta_7$  should imply  $\infty^1$  others, the number of free constants would reduce to 9:—call the number  $10 - R$ , where  $R = 0$  or 1. The third class contains 9 constants, 3 for each of the collineations involved. The use of the fourth class is probably novel, at least in this connection; it contains 3 constants. The proof of this is the essential part of the demonstration.

The argument is now most easily stated geometrically. In a linear (flat) space of 15 dimensions, two contained algebraic varieties or spreads of  $s$  and of  $k$  dimensions respectively must have in common a spread of at least  $s + k - 15$  dimensions. Here  $s = 10 - R$ ,  $k = 9$ , and the common part or intersection is of 3 dimensions. Hence  $10 - R + 9 - 15 \geq 3$ , or  $1 \geq R$ . But we had  $R \geq 1$ , therefore  $R = 1$ , as asserted in the theorem.